Proliferation law of periodic orbits of an integrable billiard as obtained from the eigenvalue spectrum

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We obtain the average proliferation law of periodic orbits for a rectangle billiard from semiclassical considerations by deriving the trace formula. We discuss that this law is the classical analog of the celebrated Weyl formula.

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Since the advent of the old quantum theory, it is well known that integrable systems can be quantized exactly even under semiclassical considerations. It is a classical problem paraphrased by Kac [1] "Can One Hear the Shape of a Drum ?" where one aims at inferring classical properties of the system (e.g., area or volume of the cavity, shape, etc.) from the eigenvalue spectrum of the system. In this Brief Report, we obtain the law of proliferation of periodic orbits for the well-studied incommensurate rectangle billiard by deriving the trace formula [2].

The spectrum of the rectangle billiard is such that the two wave numbers are $\mathbf{k_n} = (\pi n_1/a, \pi n_2/b)$, where $\mathbf{n} \in \mathbb{Z}_2$, a and b are side lengths of the rectangle with $a/b = \gamma$ an irrational number. Not only is the choice of incommensurate γ generic but also this makes the spectrum of the rectangle nondegenerate. Using periodic orbit theory, one can derive the trace formula for the rectangle billiard where one can also predict the spectral correlations in this and other more complex systems [3]. As we obtain the oscillatory correction to the density of states as a sum over periodic orbits, it can be seen [4] that the first term in the Weyl formula (the area term equal to $A_R E/4\pi$, A_R is the area of the enclosure and E is the energy) is obtained on taking the zero-length periodic orbits. There is a lot of interest on the inversion of the trace formula in general in order to obtain the length spectrum of regular and chaotic systems [5]. It is known that the results are not exact and there remains open problems. We obtain the length spectrum and hence the proliferation law for a very simple situation, significantly though we shall get an exact result. Precisely, we get the law, viz. the number of periodic orbits of length $\leq \ell$,

$$N(\ell) = \alpha\ell^2 + \beta\ell \tag{1}$$

with correct values of constants α and β. The proliferation law by itself is not an original result as we have obtained this law from number-theoretic considerations [6] with $\alpha = \pi/16ab$ and $\beta = \pi(a+b)/4ab$.

Let us consider the lattice in the $\mathbf{k}$ space, periodicities are π/a and π/b. Thus the periodicities imposed on the real space are $2a$ and $2b$. The lattice formed in the real space can be used to ascertain the density of periodic orbits. Of course, there will be a term contributing from the full lattice and the other two from the axes when one of the periodicities is present. Mathematically then, the correct density of corresponding to the quadrant in real space can be written as:

$$4N(\ell) = \sum_{all \mathbf{m}} \delta(\ell - \ell_{\mathbf{m}}) + \sum_{m_2, m_1 = 0} \delta(\ell - \ell_{\mathbf{m}}) + \sum_{m_1, m_2 = 0} \delta(\ell - \ell_{\mathbf{m}}). \tag{2}$$

Note that although the axes terms (i.e., when m_1 or m_2 is zero) are contained in the first term, we must add their contribution again as they are topologically distinctly different. Thus, taking ξ as a continuous variable,

$$4N(\ell) = \frac{\pi^2}{ab} \int d^2\xi \delta(\ell - \ell_\xi) \sum_{\mathbf{n}} e^{i\xi_1 \pi n_1/a + i\xi_2 \pi n_2/b}, \tag{3}$$

where (ξ_1, ξ_2) are the two components of the vector towards the lattice point in the direct space. From individual wave vectors, the contribution from the the first term is

$$4N_{0\mathbf{n}}(\ell) = \frac{\pi^2}{ab} \int d^2\xi \delta(\ell - \ell_\xi) e^{i\boldsymbol{\xi}\cdot\mathbf{k_n}}. \tag{4}$$

Using now the polar representation of $\boldsymbol{\xi}$, $|\mathbf{k_n}| = k_{\mathbf{n}}$ and the well-known identity [7]

$$e^{i\ell k_{\mathbf{n}} \cos\theta} = \sum_{m=-\infty}^{\infty} i^m e^{im\theta} J_m(\ell k_{\mathbf{n}}) \tag{5}$$

in

$$N_{0\mathbf{n}}(\ell) = \frac{\pi^2}{4ab} \oint d\theta \ell e^{i\ell k_{\mathbf{n}} \cos\theta}, \tag{6}$$

we get on similar contributions from other wave vectors the following:

$$N_0(\ell) = \frac{\pi}{8ab} \ell \sum_{\mathbf{n}} J_0(\ell k_{\mathbf{n}}). \tag{7}$$

On integration over ℓ, we get the cumulative length spectrum from the first term of the energy density,

$$F_0(\ell) = \frac{\pi}{8ab} \ell \sum_{\mathbf{n}} k_{\mathbf{n}}^{-1} J_1(\ell k_{\mathbf{n}}). \tag{8}$$

The small $k_{\mathbf{n}}$ corresponds to large length scales and hence this region of k is expected to give at least the average

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proliferation law which will be the dominant term, using the fact that $J_1(x) \sim x/2$ for small x, we get exactly the quadratic term in the proliferation law. However, we must also estimate the expression if the whole lattice of wave numbers is accounted for. This can be done by converting the double summation into an integral and going over to polar representation in the direct space — we get exactly 1/4 which is obviously very small as compared to the quadratic term. In a similar way, the axis terms can be calculated, we now present the full expression for the proliferation law as obtained from the lattice of wave vectors for the incommensurate rectangle billiard,

$$F(\ell) = \frac{\pi}{16ab}\ell^2 + \frac{\pi(a+b)}{ab}\ell + \frac{1}{4}. \tag{9}$$

A significant point that we wish to stress here is that, proceeding in the same way starting from the periodic orbits, one gets the Weyl formula, as is well known. Thus, in this sense, we have brought out an interesting fact — the average proliferation law is the classical analog of the Weyl formula. The inversion could be effectively carried out because the rates of growth of lattice points in direct and reciprocal space is exactly the same for rectangle billiard (we believe this to be the case for all the integrable billiards). Moreover, in chaotic systems, the inversion cannot be carried out explicitly *because* the Weyl formula is always algebraic (e.g., quadratic for the two-dimensional billiards) but the proliferation law is an exponential one.

Owing to our previous study [6], we know that the law of proliferation for pseudointegrable billiards (for every class η) of periodic orbits is also quadratic, we believe that the inversion can be carried out exactly even for these systems — the crux lies in the possibility of enumeration and classification of periodic orbits by replication and stacking technique pioneered in Ref. [8].

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